Recursion relations for two-loop self-energy diagrams on shell.

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Abstract

A set of recurrence relations for on-shell two-loop self-energy diagrams with one mass is presented, which allows to reduce the diagrams with arbitrary indices (powers of scalar propagators) to a set of the master integrals. The SHELL2 package is used for the calculation of special types of diagrams. A method of calculation of higher order ε-expansion of master integrals is demonstrated.

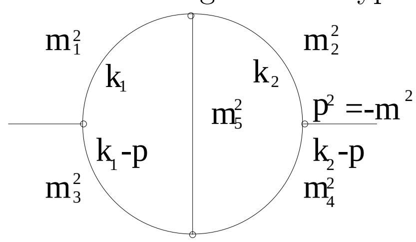
Nowadays e^+e^--experiments are sensitive to multiloop radiative corrections. The transverse part of renormalized gauge boson self-energies on mass shell enter a wide class of low energy observables like $\Delta\rho, \Delta r$, etc. Keeping in mind this physical application we elaborate a FORM [1] based package [1] [3] for analytical calculations of on-shell two-loop self-energy diagrams with one mass [2]. All possible diagrams of given type are shown on Fig.1. The diagrams implemented in the package SHELL2 [5] (**ON3**, **ON2** in our notations) and those considered in detail in Ref.[6] (**F00000**, **V0000**, **J001**, **J000**) are not discussed here. In contrary to Ref.[7] we used only recurrence relations obtained from the integration by part method [6] without shifting the dimension of space-time. Here we apply the triangle rule for arbitrary masses and external momentum given in Ref.[8]. We are working in Euclidean space-time with dimension $N = 4-2\varepsilon$. The general prototype involves arbitrary integer powers of the scalar denominators $c_L = k_L^2 + m_L^2$. For completeness we write below our notations: $c_1 = k_1^2+m_1^2$, $c_2 = k_2^2+m_2^2$, $c_3 = (k_1-p)^2+m_3^2$, $c_4 = (k_2-p)^2+m_4^4$, $c_5 = (k_1-k_2)^2+m_5^2$. Their powers j_L are called indices of the lines. The mass-shell condition for the external momentum now is $p^2 = -m^2$. Any scalar products of the momenta in the numerator are reduced to powers

[1]For a review of existing packages see Ref.[2].

[2]This package can be used also in asymptotic expansion, see, e.g.[4].

of the scalar propagators (in case of V and J topologies the corresponding lines are added). Thus, the indices may sometimes become negative. The recurrence relations allow to reduce all lines with negative indices to zero and the positive indices to one or zero.

F-topology. Let us consider the diagram of F-type:



The full set of recurrence relations valid for arbitrary $p^2, m_1^2, m_2^2, m_3^2, m_4^2, m_5^2$ is the following:

$$\{135\} \qquad N - 2j_1 - j_3 - j_5 + \frac{j_1}{c_1} 2m_1^2 + \frac{j_3}{c_3}\left(m_3^2 + m_1^2 - m^2 - c_1\right) + \frac{j_5}{c_5}\left(m_5^2 + m_1^2 - m_2^2 + c_2 - c_1\right) = 0,$$

$$\{315\} \qquad N - 2j_3 - j_1 - j_5 + \frac{j_3}{c_3} 2m_3^2 + \frac{j_1}{c_1}\left(m_3^2 + m_1^2 - m^2 - c_3\right) + \frac{j_5}{c_5}\left(m_5^2 + m_3^2 - m_4^2 + c_4 - c_3\right) = 0,$$

$$\{513\} \qquad N - 2j_5 - j_1 - j_3 + \frac{j_5}{c_5} 2m_5^2 + \frac{j_1}{c_1}\left(m_5^2 + m_1^2 - m_2^2 + c_2 - c_5\right) + \frac{j_3}{c_3}\left(m_5^2 + m_3^2 - m_4^2 + c_4 - c_5\right) = 0,$$

$$\{245\} \qquad N - 2j_2 - j_4 - j_5 + \frac{j_2}{c_2} 2m_2^2 + \frac{j_4}{c_4}\left(m_4^2 + m_2^2 - m^2 - c_2\right) + \frac{j_5}{c_5}\left(m_5^2 + m_2^2 - m_1^2 + c_1 - c_2\right) = 0,$$

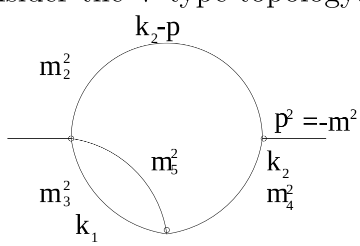
$$\{425\} \qquad N - 2j_4 - j_2 - j_5 + \frac{j_4}{c_4} 2m_4^2 + \frac{j_2}{c_2}\left(m_4^2 + m_2^2 - m^2 - c_4\right) + \frac{j_5}{c_5}\left(m_5^2 + m_4^2 - m_3^2 + c_3 - c_4\right) = 0,$$

$$\{524\} \qquad N - 2j_5 - j_2 - j_4 + \frac{j_5}{c_5} 2m_5^2 + \frac{j_2}{c_2}\left(m_5^2 + m_2^2 - m_1^2 + c_1 - c_5\right)$$

$$+\frac{j_4}{c_4}\left(m_5^2+m_4^2-m_3^2+c_3-c_5\right)=0,$$

where both sides of these relations are understood to be multiplied by $\int \frac{d^N\ k_1 d^N k_2}{c_1^{j_1} c_2^{j_2} c_3^{j_3} c_4^{j_4} c_5^{j_5}}$. Due to the symmetry of the diagram it is sufficient to consider in detail only the cases $j_1 < 0$ $\quad(c_1^{|j_1|}$ in the numerator) and $j_5 < 0$. To exclude the numerator in the first case, we apply the following relations. If $j_5 \neq 1$ we solve Eq.{245} with respect to the $\frac{j_5}{c_5}c_1$ term. If $j_2 \neq 1$ the $\frac{j_2}{c_2}c_1$ term of {524} is used. For $j_3 \neq 1$ the linear combination {245} + {135} is solved with respect to $\frac{j_3}{c_3}c_1$. In case $j_2 = j_3 = j_5 = 1$ we apply {315}, solved with respect to the free term $(N-3-j_1)$, to create a denominator for which the above given relations are applicable. The case $j_5 < 0$ is considered in detail in Refs.[6, 7, 9]. In this manner F-type integrals with **arbitrary** indices are reduced to F-type integrals with only **positive** indices or V-type integrals with **arbitrary** indices. For the former case the solution of recurrence relations is presented in Ref.[3]. Only eight diagrams **F11111, F00111, F10101, F10110, F01100, F00101, F10100, F00001** with all indices equal to 1 form the F-type basis.

V-topology. Let us consider the V-type topology:



The set of recurrence relations, valid for arbitrary mass and momenta, consist of relation {425} and the following ones:

$$\{423\} \qquad N-2j_4-j_2-j_3+\frac{j_4}{c_4}2m_4^2+\frac{j_2}{c_2}\left(m_4^2+m_2^2-m^2-c_4\right)$$
$$+\frac{j_3}{c_3}\left(m_3^2+m_4^2-m_5^2+c_5-c_4\right)=0,$$

$$\{530\} \qquad N-2j_5-j_3+\frac{j_5}{c_5}2m_5^2+\frac{j_3}{c_3}\left(m_3^2+m_5^2-m_4^2+c_4-c_5\right)=0,$$

$$\{350\} \qquad N-2j_3-j_5+\frac{j_3}{c_3}2m_3^2+\frac{j_5}{c_5}\left(m_5^2+m_3^2-m_4^2+c_4-c_3\right)=0,$$

$$
\begin{aligned}
\{A\} \qquad & \left(\frac{j_3}{c_3}+\frac{j_5}{c_5}\right)[\mathbf{NUM}]-\frac{j_5}{c_5}\left(m_4^2-m_2^2+m^2+c_2-c_4\right)=0, \\
\{B\} \qquad & \left(\frac{j_2}{c_2}+\frac{j_4}{c_4}+\frac{j_5}{c_5}\right)\left(m_4^2-m_2^2+m^2+c_2-c_4\right) \\
& -\frac{j_5}{c_5}[\mathbf{NUM}]-\frac{j_2}{c_2}2m^2=0, \\
\{C\} \qquad & \frac{j_2}{c_2}[\mathbf{NUM}]+\left(\frac{j_2}{c_2}+\frac{j_4}{c_4}+\frac{j_5}{c_5}\right)\left(m_5^2-m_3^2-m_4^2+c_3+c_4-c_5\right) \\
& -2\frac{j_5}{c_5}\left(c_3-m^2\right)=0,
\end{aligned}
$$

where we introduce

$$
[\mathbf{NUM}] \equiv \frac{(m_4^2-m_2^2+m^2+c_2-c_4)\,(m_5^2-m_3^2-m_4^2+c_3+c_4-c_5)}{2c_6},
$$

$c_6 = c_4 - m_4^2$ and the zero-index in the above relations denotes lines with zero mass and zero index. Let us discuss in detail the relations $\{A, B, C\}$. Due to the presence of a four-line vertex in this case, some scalar product arising in the recurrence relations cannot be expressed as linear combination of denominators. Nevertheless this scalar product can be presented as a nonlinear combination of a denominator and a 'new' massless propagator (see Ref.[10]). Let us consider the following distribution of momenta: $c_2 = (k_2+p)^2+m_2^2; c_3 = k_1^2+m_3^2; c_4 = k_2^2+m_4^4; c_5 = (k_1-k_2)^2+m_5^2$. Then relation $\{B\}$ reads

$$
0 \equiv \int d^N k_2 \frac{\partial}{\partial k_2^\mu}\left\{\frac{p^\mu}{c_2^{j_2}c_4^{j_4}c_5^{j_5}}\right\} \to 2m^2\frac{j_2}{c_2} - 2\frac{j_5}{c_5}k_1p - \left(\frac{j_2}{c_2}+\frac{j_4}{c_4}+\frac{j_5}{c_5}\right)k_2p = 0.
$$

The scalar product k_2p is rewritten in the following way: $k_2p = c_2 - c_4 + m_4 - m_2 + m^2$, whereas k_1p can be presented by means of the projection operator: $k_1p = A(k_1,p,k_2) + \frac{(k_1k_2)(pk_2)}{k_2^2}$, where $A(q,r,p,) = q^\mu\left(\delta_{\mu\nu} - \frac{{}_\mu p_\nu}{p^2}\right)r^\nu$ satisfies the property that odd power of $A(q,r,p)$ drop out after integration. Due to this property we have $k_1p = \frac{\left(m_4^2-m_2^2+m^2+c_2-c_4\right)\left(m_5^2-m_3^2-m_4^2+c_3+c_4-c_5\right)}{4c_6}$, where $c_6 = c_4 - m_4^2$. If $m_4^2 \neq 0$, the expression $\frac{1}{c_4c_6}$ can be simplified by partial fraction decomposition.

Numerator for V-topology. For j_2, j_3 or $j_5 < 0$ the initial diagram can be reduced to a two-loop tadpole-like integral by Eq.(2.10) in [11]. For $j_4 < 0$

we apply the following relations. If $j_5 \neq 1$ we solve Eq.{350} with respect to the $\frac{j_5}{c_5}c_4$ term. If $j_3 \neq 1$ the $\frac{j_3}{c_3}c_4$ term of {530} is used. For $j_2 \neq 1$ the linear combination {425} + {350} is solved with respect to $\frac{j_2}{c_2}c_4$. The case $j_2 = j_3 = j_5 = 1$ requires additional consideration. We distinguish the following cases for **on-shell** integrals:

1. $m_3^2 = m_5^2 = m^2, \qquad m_2^2 = 0.$

$$N - 3j_5 = \frac{j_2}{c_2}(c_5 - c_3)\left(\frac{m^2}{c_4} + 1\right) + \frac{j_5}{c_5}(c_3 - c_4) + (j_2 + 2j_4)\frac{c_5 - c_3}{c_4} - \frac{j_5}{c_5}4m^2.$$

2. $m_2^2 = m_3^2 = m^2, \qquad m_5^2 = 0.$

$$N - 3j_2 = \frac{j_5}{c_5}\frac{c_2}{c_4}m^2 - \frac{j_2}{c_2}\left(4m^2 + c_4\right) + (j_5 + 2j_4)\frac{c_2}{c_4} + \frac{j_5}{c_5}\frac{c_2}{c_4}(c_4 - c_3).$$

The other cases have been consider in Ref.[5]. In this manner the V-type integrals with **arbitrary** indices are reduces to V-type integral with only **positive** indices or J-type integral with **arbitrary** indices. For the former case the solution of recurrence relations is presented in Ref.[3]. The complete set of basic integrals is just given by **V1111, V1001** with all indices equal to 1.

J-topology. The integrals of this type are discussed in detail in Ref.[7]. We only mention here that to reduce the numerator, Eq.(7) of [12] is used. The master integrals are the following: one prototype **J111** with all indices equal to 1, and two integrals of **J011**-type: with indices 111 and 112, respectively.

Master-integrals. To obtain the finite part of two-loop physical results one needs to know the finite part of the F-type integrals, V- and J-type integrals up to the ε-part, and one-loop integrals up to the ε^2-part. The detailed discussion of the calculation of the master-integrals up to the needed order and a comparison with earlier existing results is given in Ref.[13]. Here we present the result of numerical investigations of the integral **J011**$(1,2,2,m)$. The main idea is very simple [14]: knowledge of a high precision numerical value of the integral and a set of basic irrational numbers allows to find the analytical result by applying the FORTRAN program **PSLQ** [15]. Inspired by this idea we found the next several orders of the ε-expansion of the above integral. High precision numerical results for diagrams with smallest threshold far from their mass shell (e.g. **F11111**,**V1111**, **J111**, **J011**) can be

obtained by the method elaborated in [16]: Padé approximants are calculated from the small momentum Taylor expansion of the diagram. The main problem in this procedure is to find the proper basis. At the present moment we don't know the general solution of this. Nevertheless, for **J011**(1,2,2,m) diagram we guessed the basis up to $\mathcal{O}(\varepsilon^5)$ with the following the result:

$$m^2\mathbf{J011}(1,2,2,m) = \frac{2}{3}\zeta_2 - \varepsilon\frac{2}{3}\zeta_3 + \varepsilon^2 3\zeta_4 - \varepsilon^3\left\{2\zeta_5 + \frac{4}{3}\zeta_2\zeta_3\right\}$$
$$+\varepsilon^4\left\{\frac{61}{6}\zeta_6 + \frac{2}{3}\zeta_3^2\right\} - \varepsilon^5\left\{6\zeta_7 + 4\zeta_2\zeta_5 + 6\zeta_3\zeta_4\right\} + \mathcal{O}(\varepsilon^6), \qquad (1)$$

where the general factor is $\frac{\Gamma^2(1+\varepsilon)}{(4\pi)^{\frac{N}{2}}(m^2)^{2\varepsilon}}$ is assumed. The $\mathcal{O}(\varepsilon^6)$ term is not expressible in terms of ζ-function $(\zeta_8, \zeta_3\zeta_5)$ only, so that a new irrational, like $\zeta(5,3)$ [17] e.g., may arise.

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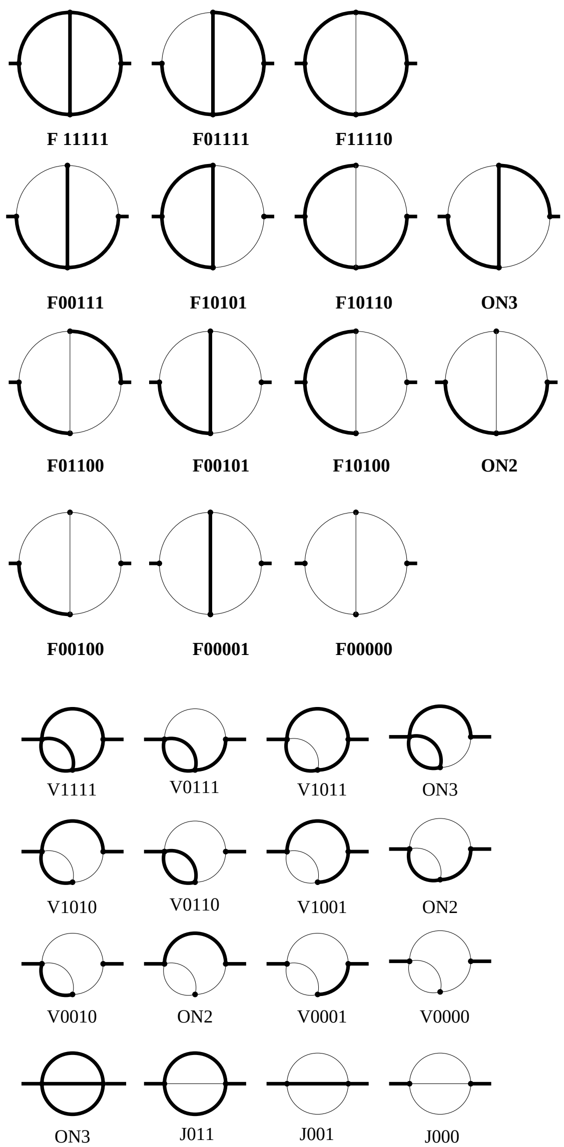

Figure 1: The F, V and J topologies. Bold and thin lines correspond to the mass and massless propagators, respectively. ON3 and ON2 are diagrams calculable by package SHELL2